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Magnetized Stiff Fluid Cylindrically Symmetric Universe with Two Degrees of Freedom in General Relativity

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Abstract A magnetized stiff fluid cylindrically symmetric universe with two degrees of freedom for perfect fluid distribution, is investigated. The magnetic field is due to an electric current produced along x-axis. The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity. The behaviour of the model in presence and absence of magnetic field is discussed. The other physical aspects of the model related to the observations are also discussed.

Keywords Magnetized · Stiff fluid · Cylindrically symmetric · Two degrees of freedom

1 Introduction

The investigations of stiff fluid universes are interesting in the sense that for such models, the velocity of sound is equal to the velocity of light and its governing equations have the same characteristics as those of gravitational field (Zel'dovich [1, 2]). The relevance of the stiff equation of state ($\rho = p$) to the matter content of the universe in its early stages has been discussed by Barrow [3]. Letelier and Tabensky [4] have investigated cylindrical self-gravitating fluid for stiff matter. Tabensky and Taub [5] have studied plane symmetric self-gravitating fluids with pressure equal to energy density. A plane symmetric universe filled with disordered radiation has been obtained by Roy and Singh [6]. Wesson [7] has obtained an exact solution to Einstein field equation with a stiff equation of state. McIntosh [8] has studied self similar cosmologies with equation of state ($p = \epsilon$). Mohanty et al. [9] have investigated cylindrically symmetric Zel'dovich fluid distribution in General Relativity. Götz [10] has obtained a plane symmetric solution of Einstein's field equation for stiff perfect fluid distribution. The large-scale intergalactic magnetic field speculated by Asseo

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and Sol [11] is of primordial origin at present measures 10^{-8} G and gives rise to a density of the order of 10^{-35} g cm^{-3}. Although, the present day magnitude of the magnetic energy is very small in comparison with the estimated matter density, it might not have been negligible during early stages of the universe. It is therefore interesting to investigate cosmological models which include an incident magnetic field to represent the early universe. The homogeneous models thus formed are necessarily anisotropic because the incident magnetic field gives rise to a preferred spatial direction and hence breaks the isotropy (Bronnikov et al. [12]). The existence of a of a primordial magnetic field is limited to Bianchi types I, II, III, VI_0 and VII_0 as shown by Hughston and Jacobs [13]. A detailed discussion of the primordial magnetic field in the case of Bianchi Type I cosmological models has been given by Thorne [14]. Roy and Singh [15] have investigated Bianchi Type V universe with stiff fluid and electromagnetic radiation. Bali and Tyagi [16] have investigated Bianchi Type I stiff magnetofluid cosmological model in General Relativity for the perfect fluid of infinite electrical conductivity. FRW models are approximately valid as present day magnetic field is very small. Maartens [17] in his study has explained that Magnetic fields are observed not only in stars but also in galaxies. In principle, these fields could play a significant role in structure formation but also affect the anisotropies in Cosmic Microwave Background radiation (CMB). Jacobs [18, 19] investigated Bianchi Type I cosmological models with magnetic field satisfying barotropic equation of state. Collins [20] has given a qualitative analysis of Bianchi Type I models with magnetic field. Roy and Prakash [21] have investigated a plane symmetric cosmological model with an incident magnetic field for perfect fluid distribution in which the gravitational field is Petrov Type I degenerate. Roy et al. [22] investigated a cylindrically symmetric universe with two degrees of freedom in General Relativity in which the gravitational fields Petrov Type I degenerate. Bali and Ali [23] have investigated a magnetized cylindrically symmetric universe with two degrees of freedom in which the gravitational field is Petrov Type I degenerate.

In this paper, we have investigated a realistic magnetized stiff fluid cylindrically symmetric universe with two degrees of freedom for perfect fluid distribution in General Relativity. The distribution consists of an electrically neutral perfect fluid with an infinite electrical conductivity. The magnetic field is due to an electric current produced along the x-axis. The physical and geometrical aspects of the model related to the observations are discussed. The behaviour of the model in absence of magnetic field related to the observations are also discussed.

We consider an anisotropic homogeneous universe with two degrees of freedom in the form given by Stachel [24] as

$$ds^2 = A^2(dx^2 - dt^2) + C^2dz^2 + (B^2 + D^2)dy^2 + 2CDdydz \tag{1.1}$$

where metric potentials A, B, C, D are functions of time alone. The energy momentum tensor T_i^j is taken into the form

$$T_i^j = (\epsilon + p)v_i v^j + pg_i^j + E_i^j \tag{1.2}$$

where E_i^j is the electromagnetic field tensor given by Lichnerowicz [25] as

$$E_i^j = \bar{\mu}\left[|h|^2\left(v_i v^j + \frac{1}{2}g_i^j\right) - h_i h^j\right] \tag{1.3}$$

ϵ is the density, p the pressure and v^i the flow vector satisfying

$$g_{ij}v^i v_j = -1 \tag{1.4}$$

$\bar{\mu}$ is the magnetic permeability and h_i the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \epsilon_{ijkl} F^{kl} v^j \tag{1.5}$$

where F^{kl} is the electromagnetic field tensor and ϵ_{ijkl} is Levi-Civita tensor density. We assume the coordinates to be commoving so that $v^1 = 0 = v^2 = v^3$ and $v^4 = \frac{1}{A}$. We also assume that uniform current is flowing along x-axis so magnetic field is in the yz-plane. Thus $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. This leads to $F_{12} = F_{13} = 0$ by virtue of (1.5). We also find that $F_{14} = 0 = F_{24} = F_{34}$ due to the assumption of infinite conductivity of the fluid Maartens [16]. A cosmological model which contains a global magnetic field is necessarily anisotropic since the magnetic field vector specifies a preferred spatial direction (Bronnikov et al. [12]). Thus only non-vanishing component of F_{ij} is F_{23}. The Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0, \quad F^{ij}_{;j} = 0 \tag{1.6}$$

are satisfied for $F_{23} = H$ (constant). Hence

$$h_1 = \frac{AH}{\bar{\mu} BC} \tag{1.7}$$

where a semi colon stands for covariant differentiation.

The Einstein field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} \tag{1.8}$$

for the line-element (1.1) lead to

$$\frac{1}{A^2}\left[\frac{A_4 B_4}{AB} + \frac{A_4 C_4}{AC} - \frac{B_4 C_4}{BC} - \frac{B_{44}}{B} - \frac{C_{44}}{C}\right] - \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 = 8\pi\left(p - \frac{H^2}{2\bar{\mu} B^2 C^2}\right) \tag{1.9}$$

$$-\frac{1}{A^2}\left[\frac{A_{44}}{A} + \frac{B_{44}}{B} - \frac{A_4^2}{A^2}\right] - \frac{1}{2}\frac{DC}{A^2 B^2}\left(\frac{D}{C}\right)_{44} + \frac{1}{2}\frac{DC}{A^2 B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B} - \frac{3C_4}{C}\right) - \frac{3}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 = 8\pi\left(p + \frac{H^2}{2\bar{\mu} B^2 C^2}\right) \tag{1.10}$$

$$-\frac{1}{A^2}\left[\frac{A_{44}}{A} + \frac{C_{44}}{C} - \frac{A_4^2}{A^2}\right] + \frac{1}{2}\frac{DC}{A^2 B^2}\left(\frac{D}{C}\right)_{44} - \frac{1}{2}\frac{DC}{A^2 B^2}\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B} - \frac{3C_4}{C}\right) + \frac{1}{4}\left[\frac{CD_4 - DC_4}{ABC}\right]^2 = 8\pi\left(p + \frac{H^2}{2\bar{\mu} B^2 C^2}\right) \tag{1.11}$$

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}\right]-\frac{1}{4}\left[\frac{CD_4-DC_4}{ABC}\right]^2$$
$$=8\pi\left(\epsilon+\frac{H^2}{2\bar{\mu}B^2C^2}\right) \tag{1.12}$$

and

$$\left(\frac{D}{C}\right)_{44}-\left(\frac{D}{C}\right)_4\left(\frac{B_4}{B}-\frac{3C_4}{C}\right)=0 \tag{1.13}$$

The suffix 4 after A, B, C and D denotes ordinary differentiation with respect to t. We assume that the model is filled with a stiff fluid of perfect fluid distributions so we have $\epsilon = p$. Then from (1.9) and (1.13), we have

$$\frac{2B_4C_4}{BC}+\frac{B_{44}}{B}+\frac{C_{44}}{C}=\frac{8\pi H^2A^2}{\bar{\mu}B^2C^2} \tag{1.14}$$

From (1.9), (1.10) and (1.11), we have

$$\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)-\frac{B_4C_4}{BC}-\frac{B_{44}}{B}-\frac{C_{44}}{C}+\frac{1}{2}\left[\frac{CD_4-DC_4}{BC}\right]^2$$
$$+\frac{A_{44}}{A}+\frac{B_{44}}{B}-\frac{A_4^2}{A^2}+\frac{8\pi H^2A^2}{\bar{\mu}B^2C^2}=0 \tag{1.15}$$

and

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}=\left[\frac{CD_4-DC_4}{BC}\right]^2 \tag{1.16}$$

Equation (1.13) on integration leads to

$$\frac{CD_4-DC_4}{BC}=\frac{K}{C^2} \tag{1.17}$$

where K is constant of integration.

From (1.16) and (1.17), we have

$$\frac{C_{44}}{C}-\frac{B_{44}}{B}=\frac{K^2}{C^4} \tag{1.18}$$

From (1.14), (1.15) and (1.16), we have

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\frac{(BC)_4}{BC}=-\frac{1}{2}\frac{(BC)_{44}}{BC} \tag{1.19}$$

Equations (1.14) and (1.19) lead to

$$(BC)_{44}=\frac{b^2A^2}{BC} \tag{1.20}$$

and

$$\frac{A_4}{A}=-\frac{1}{2}\frac{(BC)_4}{BC}+\frac{L}{BC} \tag{1.21}$$

where $b^2 = \frac{8\pi H^2}{\bar{\mu}}$ and L is constant of integration.

Let us consider

$$BC = \mu \quad \text{and} \tag{1.22a}$$

$$\frac{B}{C} = \nu \tag{1.22b}$$

Using (1.22) in (1.20) and (1.21), we have

$$A^2 = \frac{\mu\mu_{44}}{b^2} \quad \text{and} \tag{1.23}$$

$$\frac{A_4}{A} = -\frac{1}{2}\frac{\mu_4}{\mu} + \frac{L}{\mu} \tag{1.24}$$

From (1.23) and (1.24), we have

$$\frac{\mu_{444}}{\mu_{44}} + \frac{2\mu_4}{\mu} - \frac{2L}{\mu} = 0 \tag{1.25}$$

Putting $\mu_4 = f(\mu)$ in (1.25), we have

$$\mu(ff'' + f'^2) + 2ff' - 2Lf' = 0 \tag{1.26}$$

which leads to

$$\frac{d}{d\mu}\left(\mu ff' + \frac{1}{2}f^2\right) = \frac{d}{d\mu}(2Lf) \tag{1.27}$$

which on integration leads to

$$\mu ff' + \frac{f^2}{2} = 2Lf + Q \tag{1.28}$$

where Q is constant of integration and $f' = \frac{df}{d\mu}$, $f'' = \frac{d^2f}{d\mu^2}$. From (1.28), we have

$$\mu = \frac{l\{\gamma + (f - 2L)\}^{\frac{2L}{\gamma} - 1}}{\{\gamma - (f - 2L)\}^{\frac{2L}{\gamma} + 1}} \tag{1.29}$$

where l is constant of integration and $\gamma^2 = 4L^2 + 2Q$.

Using (1.22) in (1.18), we have

$$\frac{dv}{\sqrt{M - K^2 e^{2v}}} = \frac{1}{\mu}\frac{dt}{d\mu}\frac{d\mu}{df}df = \frac{df}{\mu ff'} \tag{1.30}$$

where M is constant of integration. Equation (1.30) after using (1.28), leads to

$$e^v = \frac{\sqrt{M}}{K}\operatorname{sech}\phi \tag{1.31}$$

where $\phi = \log[N^\gamma\{\frac{\gamma-(f-2L)}{\gamma+(f-2L)}\}]^{\frac{\sqrt{M}}{\gamma}}$ and N is constant of integration.

From (1.23), (1.29) and (1.31), we get

$$A^2 = \frac{\gamma^2 - (f-2L)^2}{2b^2} \tag{1.32}$$

$$B^2 = \mu\nu$$
$$= \frac{\sqrt{M}l}{K} \frac{\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \operatorname{sech}\phi \tag{1.33}$$

and

$$C^2 = \frac{\mu}{\nu}$$
$$= \frac{K}{\sqrt{M}} \frac{\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \cosh\phi \tag{1.34}$$

From (1.17), (1.29) and (1.31), we get

$$\frac{D}{C} = R - \frac{\sqrt{M}}{K} \tanh\phi \tag{1.35}$$

where R is constant of integration.

From equations (1.34) and (1.35), we have

$$D^2 = \left(R - \frac{\sqrt{M}}{K}\tanh\phi\right)^2 \frac{Kl}{\sqrt{M}} \frac{\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \cosh\phi \tag{1.36}$$

and

$$CD = \left(R - \frac{\sqrt{M}}{K}\tanh\phi\right) \frac{Kl}{\sqrt{M}} \frac{\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \cosh\phi \tag{1.37}$$

Hence, the metric (1.1) reduces to the form

$$\begin{aligned} ds^2 = {} & \frac{\gamma^2 - (f-2L)^2}{2b^2}\left(dx^2 - \frac{df^2}{f^2 f'^2}\right) + \frac{Kl}{\sqrt{M}} \frac{\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \cosh\phi\, dz^2 \\ & + \frac{l\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \left[\left(\frac{\sqrt{M}}{K} + \frac{KR^2}{\sqrt{M}}\right)\cosh\phi - 2R\sinh\phi\right] dy^2 \\ & + \frac{2l\{\gamma + (f-2L)\}^{\frac{2L}{\gamma}-1}}{\{\gamma - (f-2L)\}^{\frac{2L}{\gamma}+1}} \left(\frac{RK}{\sqrt{M}}\cosh\phi - \sinh\phi\right) dy\,dz \end{aligned} \tag{1.38}$$

By introducing the following transformations

$$
\begin{aligned}
&x = X\\
&\frac{1}{M^{1/4}\sqrt{K}}y = Y\\
&\frac{\sqrt{K}}{M^{1/4}}z = Z\\
&f - 2L = T\\
&l = 4b^{\frac{4L}{\gamma}+2}
\end{aligned}
$$

the line element (1.38) reduces to the form

$$
\begin{aligned}
ds^2 = {}& \frac{\gamma^2 - T^2}{2b^2}\left[dX^2 - 16b^{\frac{8L}{\gamma}+4}\frac{(\gamma+T)^{\frac{4L}{\gamma}-4}}{(\gamma-T)^{\frac{4L}{\gamma}+4}}dT^2\right] + \frac{(\gamma+T)^{\frac{2L}{\gamma}-1}}{(\gamma-T)^{\frac{2L}{\gamma}+1}}\left(4b^{\frac{4L}{\gamma}+2}\right)\cosh\phi\, dZ^2\\
&+ \frac{(\gamma+T)^{\frac{2L}{\gamma}-1}}{(\gamma-T)^{\frac{2L}{\gamma}+1}}\left(4b^{\frac{4L}{\gamma}+2}\right)\{(\beta^2+\alpha^2)\cosh\phi - 2\beta\alpha\sinh\phi\}dY^2\\
&+ \frac{2(\gamma+T)^{\frac{2L}{\gamma}-1}}{(\gamma-T)^{\frac{2L}{\gamma}+1}}\left(4b^{\frac{4L}{\gamma}+2}\right)(\alpha\cosh\phi - \beta\sinh\phi)dY\,dZ
\end{aligned} \tag{1.39}
$$

where

$$
\beta = \sqrt{M}, \quad \alpha = KR \quad \text{and} \quad \phi = \sqrt{M}\log\left\{N\left(\frac{\gamma-T}{\gamma+T}\right)^{\frac{1}{\gamma}}\right\}
$$

By the transformation

$$
T = \gamma\cos 2b\tau \tag{1.40}
$$

the line element (1.39) reduces to

$$
\begin{aligned}
ds^2 = {}& \frac{\gamma^2\sin^2 2b\tau}{2b^2}\left[dX^2 - 16b^{\frac{8L}{\gamma}+4}\left(\frac{1+\cos 2b\tau}{1-\cos 2b\tau}\right)^{\frac{4L}{\gamma}}\frac{1}{\gamma^8}\frac{1}{\sin^8 2b\tau}d\tau^2\right]\\
&+ \left(\frac{1+\cos 2b\tau}{1-\cos 2b\tau}\right)^{\frac{2L}{\gamma}}\frac{1}{\gamma^2\sin^2 2b\tau}4b^{\frac{4L}{\gamma}+2}\cosh\phi\, dZ^2\\
&+ \left(\frac{1+\cos 2b\tau}{1-\cos 2b\tau}\right)^{\frac{2L}{\gamma}}\frac{1}{\gamma^2\sin^2 2b\tau}4b^{\frac{4L}{\gamma}+2}\{(\beta^2+\alpha^2)\cosh\phi - 2\beta\alpha\sinh\phi\}dY^2\\
&+ 2\left(\frac{1+\cos 2b\tau}{1-\cos 2b\tau}\right)^{\frac{2L}{\gamma}}\frac{1}{\gamma^2\sin^2 2b\tau}4b^{\frac{4L}{\gamma}+2}(\alpha\cosh\phi - \beta\sinh\phi)dY\,dZ
\end{aligned} \tag{1.41}
$$

In absence of magnetic field, the line element (1.41) reduces to the form

$$
ds^2 = 2\gamma^2\tau^2\left[dX^2 - \frac{1}{\gamma^6\tau^{\frac{8L}{\gamma}+6}}d\tau^2\right] + \frac{1}{\gamma^2\tau^{\frac{4L}{\gamma}+2}}\cosh\phi\, dZ^2
$$

$$+\frac{1}{\gamma^2\tau^{\frac{4L}{\gamma}+2}}\{(\alpha^2+\beta^2)\cosh\phi-2\alpha\beta\sinh\phi\}dY^2$$
$$+\frac{2}{\gamma^2\tau^{\frac{4L}{\gamma}+2}}(\alpha\cosh\phi-\beta\sinh\phi)dYdZ \tag{1.42}$$

2 Some Physical and Geometrical Features

The density and pressure for the model (1.39) are given by

$$\begin{aligned}8\pi\epsilon &= 8\pi p\\ &= \frac{(\gamma-T)^{\frac{4L}{\gamma}+2}}{16b^{\frac{8L}{\gamma}+4}(\gamma+T)^{\frac{4L}{\gamma}-2}}\\ &\quad\times\left[\frac{(4L^2-T^2-\beta^2\tanh^2\phi)}{32b^{\frac{8L}{\gamma}+2}}.\frac{(\gamma-T)^{\frac{4L}{\gamma}+1}}{(\gamma+T)^{\frac{4L}{\gamma}-1}}-4\beta^2b^{\frac{8L}{\gamma}+4}\operatorname{sech}^2\phi-\frac{b^2}{2}\right]\end{aligned} \tag{2.1}$$

The model has to satisfy the reality conditions given by Ellis [26] as

$$\begin{aligned}&\text{(i)}\quad \epsilon+p>0\\ &\text{(ii)}\quad \epsilon+3p>0\end{aligned} \tag{2.2}$$

These conditions together lead to

$$\frac{32b^{\frac{8L}{\gamma}+2}(4\beta^2b^{\frac{8L}{\gamma}+4}\operatorname{sech}^2\phi+\frac{b^2}{2})}{(4L^2-T^2-\beta^2\tanh^2\phi)}<\frac{(\gamma-T)^{\frac{4L}{\gamma}+1}}{(\gamma+T)^{\frac{4L}{\gamma}-1}} \tag{2.3}$$

The expansion θ calculated for the flow vector v^i is given by

$$\theta=\frac{1}{b^{\frac{4L}{\gamma}+1}}\frac{(\gamma-T)^{\frac{2L}{\gamma}+\frac{1}{2}}}{4\sqrt{2}(\gamma+T)^{\frac{2L}{\gamma}-\frac{1}{2}}}(T+4L) \tag{2.4}$$

The non-vanishing components of shear tensor σ_i^j are given by

$$\sigma_1^1=\frac{-2}{4b^{\frac{4L}{\gamma}+2}}\sqrt{\frac{2b^2}{\gamma^2-T^2}}\frac{(\gamma-T)^{\frac{2L}{\gamma}+1}}{3(\gamma+T)^{\frac{2L}{\gamma}-1}}(T+L) \tag{2.5}$$

$$\sigma_2^2=\frac{1}{4b^{\frac{4L}{\gamma}+2}}\sqrt{\frac{2b^2}{\gamma^2-T^2}}\frac{(\gamma-T)^{\frac{2L}{\gamma}+1}}{3(\gamma+T)^{\frac{2L}{\gamma}-1}}\left(L+T-\frac{3}{2}\alpha\right) \tag{2.6}$$

$$\sigma_3^3=\frac{1}{4b^{\frac{4L}{\gamma}+2}}\sqrt{\frac{2b^2}{\gamma^2-T^2}}\frac{(\gamma-T)^{\frac{2L}{\gamma}+1}}{3(\gamma+T)^{\frac{2L}{\gamma}-1}}\left(L+T+\frac{3}{2}\alpha\right) \tag{2.7}$$

$$\sigma_2^3=\frac{1}{4b^{\frac{4L}{\gamma}+2}}\sqrt{\frac{2b^2}{\gamma^2-T^2}}\frac{(\gamma-T)^{\frac{2L}{\gamma}+1}}{2(\gamma+T)^{\frac{2L}{\gamma}-1}} \tag{2.8}$$

The rotation (ω) is identically zero. Clearly $v^i_{;j}v^j=0$ so that the lines of flow is geodetic.

The non-vanishing components of conformal curvature tensor are given by

$$\begin{aligned}
C^{12}_{12} &= -C^{34}_{34} \\
&= \frac{1}{16b^{\frac{8L}{\gamma}+4}} \frac{b^2}{6(\gamma^2 - T^2)} \frac{(\gamma - T)^{\frac{4L}{\gamma}+2}}{(\gamma + T)^{\frac{4L}{\gamma}-2}} \\
&\quad \times (3T^2 - \gamma^2 + 6LT + 4L^2 - \beta^2 - 3\beta T \tanh\phi) \qquad (2.9)\\
C^{13}_{13} &= -C^{24}_{24} \\
&= \frac{1}{16b^{\frac{8L}{\gamma}+4}} \frac{b^2}{6(\gamma^2 - T^2)} \frac{(\gamma - T)^{\frac{4L}{\gamma}+2}}{(\gamma + T)^{\frac{4L}{\gamma}-2}} \\
&\quad \times (3T^2 - \gamma^2 + 6LT + 4L^2 - \beta^2 + 3\beta T \tanh\phi) \qquad (2.10)\\
C^{23}_{23} &= -C^{14}_{14} \\
&= \frac{-1}{16b^{\frac{8L}{\gamma}+4}} \frac{b^2}{6(\gamma^2 - T^2)} \frac{(\gamma - T)^{\frac{4L}{\gamma}+2}}{(\gamma + T)^{\frac{4L}{\gamma}-2}} (6T^2 - 2\gamma^2 + 12LT + 8L^2 - 2\beta^2) \qquad (2.11)\\
C^{13}_{12} &= C^{34}_{24} \\
&= \frac{1}{16b^{\frac{8L}{\gamma}+4}} \frac{b^2}{2(\gamma^2 - T^2)} \frac{(\gamma - T)^{\frac{4L}{\gamma}+2}}{(\gamma + T)^{\frac{4L}{\gamma}-2}} \beta T \tanh\phi \qquad (2.12)
\end{aligned}$$

Thus

$$C^{12}_{12} + C^{13}_{13} = C^{23}_{23} \qquad (2.13)$$

3 Discussion

The model (1.39) in the presence of magnetic field starts with a big-bang at $T = -\gamma$ when $4L - \gamma > 0$ and the expansion in the model decreases as time increases. The expansion in the model stops at $T = \gamma$ or at $T = -4L$. However, if $4L - \gamma < 0$ then the expansion in the model increases as time increases during span of time $0 < T < \gamma$. At $T = 0$, the expansion (θ) is given by $\theta = \frac{4L\gamma}{b^{\frac{4L}{\gamma}+1}} =$ constant. The role of magnetic field is to retard the expansion in the model. Since $v^i_{;j} v^j = 0$. Hence the lines of the flow is geodetic. The space-time is Petrov Type I non-degenerate in general. However, if $\beta = 0$ then space-time is Petrov Type D and for large value of T, it is conformally flat.

The ratio of magnetic energy to material energy is given by

$$\frac{E^4_4}{\epsilon} = \frac{128\pi b^2}{[\frac{(4L^2 - T^2 - \beta^2 \tanh^2\phi)}{32b^{\frac{8L}{\gamma}+2}} . \frac{(\gamma - T)^{\frac{4L}{\gamma}+1}}{(\gamma + T)^{\frac{4L}{\gamma}-1}} - 4\beta^2 b^{\frac{8L}{\gamma}+4} \operatorname{sech}^2\phi - \frac{b^2}{2}]}$$

The matter field dominates the magnetic energy at the initial singularity $T = -\gamma$ and the late time provided $4L > \gamma$. However, this ratio is non-zero finite quantity at $T = 0$ and tends to zero when $T \to \infty$. The anisotropy is constant initially but since $\lim_{T\to\infty} \frac{\sigma}{\theta} \neq 0$. Hence the model does not approach isotropy for large values of T.

The deceleration parameter (q) is given by

$$q = -\frac{\ddot{R}/R}{\dot{R}^2/R^2} = -18L(4L - T) \begin{cases} < 0 & \text{if } L(4L - T) > 0, \\ > 0 & \text{if } L(4L - T) < 0 \end{cases}$$

Thus the model (1.39) presents expanding universe if $L(4L - T) > 0$, q is positive when $4L < T$ and $L > 0$. It vanishes at $T = 4L$, q is always negative and approaches the value (-1) if $L(4L - T) = 1/18$ as in the case of de-Sitter universe. The scale factor (R) is given by

$$R^3 = 2b^{\frac{4L}{\gamma}} \frac{(\gamma - T)^{2L/\gamma}}{(\gamma - T)^{2L/\gamma}}$$

Thus R increases as T increases and $\frac{2L}{\gamma} > 0$. The model (1.39) has Point Type singularity at $T = -\gamma$ (MacCallum [27]).

In absence of magnetic field, the pressure and density for the model (1.42) are given by

$$8\pi p = \tau^{\frac{8L}{\gamma}+4} \left[\frac{(4L^2 - \gamma^2 - \beta^2 \tanh^2 \phi)\gamma^6}{8} \tau^{\frac{8L}{\gamma}+2} \right] \tag{3.2}$$

$$8\pi \in = \tau^{\frac{8L}{\gamma}+4} \left[\frac{(4L^2 - \gamma^2 - \beta^2 \tanh^2 \phi)\gamma^6}{8} \tau^{\frac{8L}{\gamma}+2} \right] \tag{3.3}$$

The expansion (θ), the non-vanishing components of shear tensor and conformal curvature tensor in absence of magnetic field are given by

$$\theta = \frac{\gamma}{2\sqrt{2}} (\gamma + 4L) \tau^{\frac{4L}{\gamma}+1} \tag{3.4}$$

$$\sigma_1^1 = \frac{-2}{3\sqrt{2}} \gamma \tau^{\frac{4L}{\gamma}+1} (\gamma + L) \tag{3.5}$$

$$\sigma_2^2 = \frac{1}{3\sqrt{2}} \gamma \tau^{\frac{4L}{\gamma}+1} \left(L + \gamma - \frac{3}{2}\alpha \right) \tag{3.6}$$

$$\sigma_3^3 = \frac{1}{3\sqrt{2}} \gamma \tau^{\frac{4L}{\gamma}+1} \left(L + \gamma + \frac{3}{2}\alpha \right) \tag{3.7}$$

$$\sigma_2^3 = \frac{1}{3\sqrt{2}} \gamma \tau^{\frac{4L}{\gamma}+1} \tag{3.8}$$

and

$$C_{12}^{12} = \frac{1}{24} \gamma^2 \tau^{\frac{8L}{\gamma}+2} (3\tau^2 - \gamma^2 + 6L\tau + 4L^2 - \beta^2 - 3\tau\beta \tanh \phi) \tag{3.9}$$

$$C_{13}^{13} = \frac{1}{24} \gamma^2 \tau^{\frac{8L}{\gamma}+2} (3\tau^2 - \gamma^2 + 6L\tau + 4L^2 - \beta^2 + 3\tau\beta \tanh \phi) \tag{3.10}$$

$$C_{23}^{23} = -\frac{1}{24} \gamma^2 \tau^{\frac{8L}{\gamma}+2} (6\tau^2 - 2\gamma^2 + 12L\tau + 8L^2 - 2\beta^2) \tag{3.11}$$

$$C_{12}^{13} = \frac{1}{24} \frac{\beta \tanh \phi}{\gamma^2 \tau} \tag{3.12}$$

The reality conditions given by Ellis [26]

(i) $\epsilon + p > 0$ and
(ii) $\epsilon + 3p > 0$

together lead to

$$\gamma^2 + \beta^2 \tanh^2\phi < 4L^2 \tag{3.13}$$

The expansion in the absence of magnetic field increases as time increases when $4L + \gamma > 0$. The model starts with a big-bang at $\tau = 0$ when $4L + \gamma < 0$ and the expansion in the model decreases as time increases. Since $\frac{\sigma}{\theta} \neq 0$. Hence the anisotropy is maintained. However the model isotropizes for large value of τ when $4L + \gamma < 0$. The reality condition (i) $\epsilon + p > 0$, (ii) $\epsilon + 3p > 0$ given by Ellis [26] are satisfied when $\gamma^2 + \beta^2 \tanh^2\phi < 4L^2$. The energy density $(\epsilon) \to \infty$ when $\tau \to 0$ and $\beta L + 3\gamma > 0$. $\epsilon \to 0$ when $\tau \to \infty$ and $\beta L + 3\gamma < 0$. Thus the model (1.42) in absence of magnetic field represents a realistic universe. The space-time (1.42) is Petrov Type I non-degenerate in general. However, if $\beta = 0$, it is Petrov Type D and it is conformally flat for large values of τ when $4L + \gamma < 0$. The model (1.42) has Point Type singularity MacCallum [27] at $\tau = 0$ when $2L + \gamma < 0$. The deceleration parameter (q) in absence of magnetic field is given by

$$q = -\frac{\ddot{R}/R}{\dot{R}^2/R^2} = -18L(4L-\gamma)\begin{cases} <0 & \text{if } L(4L-\gamma) > 0, \\ >0 & \text{if } L(4L-\gamma) < 0 \end{cases}$$

The model (1.42) is an expanding universe if $L(4L - \gamma) > 0$. q is always negative and approaches the value (-1) if $L(4L - \gamma) = \frac{1}{18}$ as in the case of de-Sitter universe. The scale factor in absence of magnetic field is given by

$$R^3 = \frac{2}{\tau^{4L/\gamma}}$$

Thus R increases at τ increases if $\frac{4L}{\gamma} < 0$. The Hubble parameter (H) in the absence of magnetic field is given by

$$\begin{aligned} H &= \frac{\dot{R}}{R} \\ &= \frac{8L\gamma^2}{3}\tau^{\frac{4L}{\gamma}+2} \end{aligned} \tag{3.14}$$

and density (ϵ) in the absence of magnetic field is given from (3.3) as

$$8\pi G\epsilon = \tau^{\frac{8L}{\gamma}+4}\left[\frac{(4L - \gamma^2 - \beta^2\tanh^2\phi)\gamma^6}{8}\tau^{\frac{8L}{\gamma}+2}\right] \tag{3.15}$$

Comparing (3.14) and (3.15), we find that

$$8\pi G\epsilon = 3H^2 \tag{3.16}$$

provided

$$\tau^{\frac{8L}{\gamma}+2} = \frac{512L^2}{3\gamma^2(4L^2 - \gamma^2 - \beta^2\tanh^2\phi)} \tag{3.17}$$

which is in agreement with Einstein-de-Sitter model for the average density of the universe. Taking the range value of H as 50–100 $\text{kms}^{-1}\,\text{Mpc}^{-1}$ $G = 6.67\times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$, we have the present value of average density (ϵ) of the universe for the Einstein-de-Sitter model ($k = 0$),

$$\epsilon = \frac{3H^2}{8\pi G} = 0.5\text{–}2\times 10^{-19}\,\text{gm}\,\text{cm}^{-3} \tag{3.18}$$

If the average density exceeds the value given by (3.18) then self gravitation of the universe will be able to arrest its expansion and eventually brings out contraction. The universe will then be finite or closed (as for $k > 0$). On the other hand, if density of the universe is less than that given by (3.18), the universe will always be expanding (as for $k < 0$) and it will then be open universe.

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